Voltage Function on the Infinite Grid and Convergence of Pseudoinverse Operators

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Abstract: The calculation method of the resistance between two points on an infinite grid has attracted many mathematicians for its promising engineering application. To calculate the resistance, the voltage function on the infinite grid should be calculated first given the current flowing in and out. The voltage function is a potential energy function, indicating that the choice of its zero point for a certain case is troublesome. In this paper, we consider the infinite network Z^2 and define the voltage function in the space $l^2(Z^2)$. Then we prove that the proposed voltage function is a pointwise limit and a limit in norm in another space $D/\mathbb{R}$ of the voltage function on a series of finite wire subgraphs, which makes its physical image clear. Based on the convergence of voltage function, we prove that the pseudoinverse operator of such square grids is convergent in strong operator topology from a certain dense subspace of $l^2(Z^2)$ to the Hilbert space $D/\mathbb{R}$.

1. Introduction

On a finite grid, voltage function determines the current flowing into a certain point and out of another one. The voltage function cannot be completely determined, because two voltage functions with constant difference produce the same current flow. In this paper, the voltage functions in $l^2(\Delta_N)$ that are orthogonal to $\hat{e}_N$ are considered. The consideration behind this is $\hat{e}_N$ is the only eigenvector of the Laplacian matrix of a finite grid with eigenvalue zero. So, on the orthocomplement of $\hat{e}_N$ the Laplacian transform that transforms the voltage function to the current function becomes invertible, and we can find the uniquely determined voltage function that produces the current flow.

Obviously $d((x_1,y_1),(x_2,y_2))=|x_1-x_2|+|y_1-y_2|$ is a metric on Z^2 and if $d((x_1,y_1),(x_2,y_2))=1$, the two points (x_1,y_1) and (x_2,y_2) are defined to be adjacent. In this paper, the network Z^2 is the graph generated by adding an edge between any two adjacent points in Z^2.It is assumed that every edge has the same unit resistance. And Δ_N is the subgraph with the vertices in $\{(x, y)\in Z^2||x|,|y|\le N\}$. For example, the diagram of the Δ_1 and Δ_2 is:

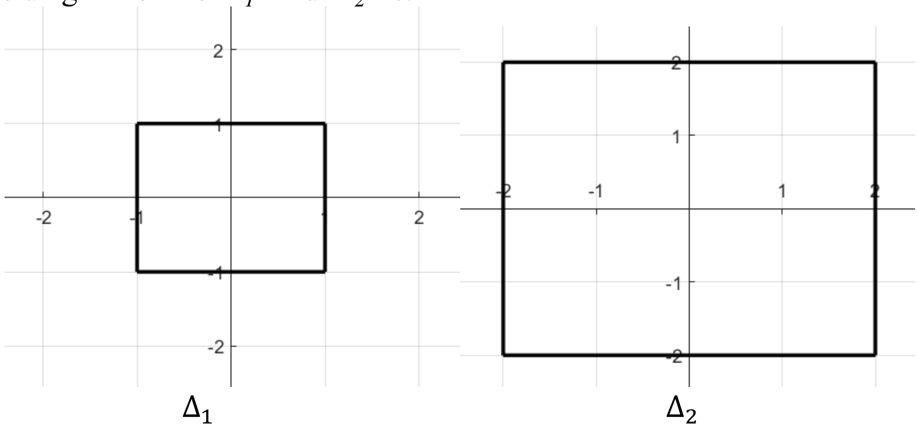

Fig.1 The Diagram of the Δ_1 and Δ_2

There are $(2N+1)^2$ points in the square Δ_N and we consider the subspace of $l^2(Z^2)$, that is $l^2(\Delta_N)$ which is made up of functions in $l^2(Z^2)$ supported in Δ_N. Let H_{Δ_N} the Laplacian matrix of graph Δ_N. Let $H^{\dagger}_{\Delta_N}$ be the pseudoinverse of H_{Δ_N}. The wired graph of Δ_N, Δ_N^w, is the graph generated by identifying all the points in Δ_N^c as one single point z_n. Let $H_{\Delta_N^w}$ the Laplacian matrix of graph Δ_N^w. Let $H^{\dagger}_{\Delta_N^w}$ be the pseudoinverse of $H_{\Delta_N^w}$.

Define a special set of functions in $l^2(Z^2)$:

$$S = \{e_a | e_a(a) = 1, e_a = 0 \text{ elsewhere}\}$$

Consider the Laplacian operator on $l^2(Z^2)$:

$$H_{Z^2}: l^2(Z^2) \to l^2(Z^2)$$
$$f \mapsto H_{Z^2}(f)$$

with

$$H_{Z^2}(f)(m) = \sum_{n\ adjacent\ to\ m} f(m) - f(n) \tag{1}$$

It can also be observed that

$$H_{\Delta_N}(f)(m) = \sum_{\substack{n\ adjacent\ to\ m \\ m,n \in \Delta_N}} f(m) - f(n) \tag{2}$$

$$H_{\Delta_N^w}(f)(m) = \sum_{\substack{n\ adjacent\ to\ m \\ m,n \in \Delta_N^w}} f(m) - f(n) \tag{3}$$

So, if there is a unit current flows into a point a and out of another point z, and on the subgraph Δ_N, the corresponding voltage on the vertices is the function $v \in l^2(\Delta_N)$, we have:

$$H_{\Delta_N}(v) = e_a - e_z$$

Define another set of current source functions $S_1 = \{f \,|\, f = e_a - e_z, \text{ for some } a \text{ and } z\}$. Any function in the set is corresponding to a unit flow. Because $e_a - e_z \in \hat{e}_N^{\perp}$, we have:

$$v = H^{\dagger}_{\Delta_N}(e_a - e_z)$$

It should be noticed that this voltage function can vary within a constant difference to produce the same current flow.

For general graphs $G = (V, E)$ where V is the set of vertices and E is the set of edges. Assume that each edge has two orientations $e, -e$ and both $e, -e \in E$ are assigned with a same resistance $r(e) \geq 0$. Let e^+ and e^- be the start point and end point of the oriented edge e. Define the Hilbert space of antisymmetric current functions on E:

$$l^2_-(E, r) = \{\theta | \theta(e) = -\theta(-e)\} \tag{4}$$

And the space has the inner product:

$$(\theta_1, \theta_2)_r = \frac{1}{2}\sum_{e \in E} r(e)\theta_1(e)\theta_2(e) \tag{5}$$

The corresponding norm is

$$||\theta_1||_r = \left(\frac{1}{2}\sum_{e \in E} r(e)\theta_1(e)^2\right)^{0.5} \tag{6}$$

Obviously, on the infinite network Z^2, the norm of a current flow generated by a voltage function is as follows:

$$||i||_r = \left(\frac{1}{2}\sum_{e\in E(Z^2)} r(e)\mathrm{i}(e)^2\right)^{0.5} = \left(\sum_{m\ and\ n\ adjacent} \big(v(m)-v(n)\big)^2\right)^{0.5}$$

Define the operator from $l^2(V)$ to $l^2_-(E, r)$:

$$d: l^2(V) \rightarrow l^2_-(E,r)$$

$$d(v)(e) = v(e^+) - v(e^-)$$

As is to be shown in section 2, random walks on a network are closely related to the voltage functions[1,2].

2. The Limit of Voltage and Current for Finite Graphs

In the following statement, v refers to voltage, i refers to current, R refers to resistance, C refers to conductance, P refers the probability function of the associated random walk.

Theorem 2.1[3]. On a finite network a current flows into a vertex a and flows out of another vertex z with unit voltage ($v(a)$=1, $v(z)$=0), then for every $x \in V(G)$, $v(x)=P_x[\tau_a<\tau_z]$.

Theorem 2.2[3]. On a finite network a current flows into a vertex a and flows out of another set of vertices Z with unit voltage ($v(A)$=1, $v(Z)$=0) then $C(a\leftrightarrow Z)=\pi(a)P_a[\tau_Z<\tau_a^+]$.

Theorem 2.3[3]. On a finite network E a current flows into a set of vertices A and flows out of another set of vertices Z with unit voltage ($v(A)$=1, $v(Z)$=0), then

$R(A\leftrightarrow Z)=\min\{||\theta||_r^2;\ \theta$ is a unit flow from A to $Z\}$

And the minimum is achieved when the flow equals to the current flow i from A to Z, and for any flow θ, $||\theta||_r^2=||\theta-i||_r^2+||i||_r^2$

Theorem 2.4[3]. G_n is an exhaustion of a recurrent network and G_n^W is the graph obtained from G by identifying the vertices outside G_n to a single vertex z_n. Then for any fixed vertex a that lies in every finite graph G_n, $R(a\leftrightarrow z_n)\rightarrow\infty$.

Theorem 2.5[3]. Let A and Z be two sets of vertices in a finite network, then for any vertex $x\notin A\cup Z$, we have

$$P_x[\tau_A < \tau_Z] \le \frac{C(x\leftrightarrow A)}{C(x\leftrightarrow A\cup Z)} = \frac{R(x\leftrightarrow A\cup Z)}{R(x\leftrightarrow A)} \tag{7}$$

Theorem 2.6[3]: G is a recurrent network with an exhaustion by subnetworks G_n and a, $z\in V(G_n)$ for all n. Let v_n be the voltage function on G_n that induced by a unit voltage at a and 0 voltage at z. Let i_n be the unit current flow on G_n from a to z, then:

(a) $v=\lim_{n\to\infty} v_n$ exists pointwise and that $v(x)=P_x[\tau_a<\tau_z]$ for all $x \in V$.

(b) $i=\lim_{n\to\infty} i_n$ exists pointwise and the sequence converges in norm.

(c) $\varepsilon(i)dv=ir$.

Proof:

(a) Due to theorem 1, $v_n(x)=P_x^n[\tau_a<\tau_z]$, here P_x^n denotes the probability function of random walk on the finite graph G_n started at vertex x. Now consider two random walks, X on G_n and Y on G starting at x, and $\tau=\min\{t|Y_t\notin G_n\}$ is the first escaping time of Y. Let $X_t=Y_t(t<\tau)$ and after τ let X be a random walk on G_n. Then we have:

$$v_n(x) = P_X[\tau_a < \tau_z]$$

$$v(x) = P_Y[\tau_a < \tau_z]$$

And

$$P_Y[\tau_a < \tau_z] = P_Y[\tau_a < \tau_z < \tau] + P_Y[\tau_a < \tau < \tau_z] + P_Y[\tau < \tau_a < \tau_z]$$
$$P_X[\tau_a < \tau_z] = P_X[\tau_a < \tau_z < \tau] + P_X[\tau_a < \tau < \tau_z] + P_X[\tau < \tau_a < \tau_z]$$

Notice that

$$P_Y[\tau_a < \tau_z < \tau] = P_X[\tau_a < \tau_z < \tau]$$
$$P_Y[\tau_a < \tau < \tau_z] = P_X[\tau_a < \tau < \tau_z]$$

So

$$|v(x) - v_n(x)| \leq P_Y[\tau < \tau_a < \tau_z] + P_X[\tau < \tau_a < \tau_z] \leq 2P_Y[\tau < \tau_a, \tau < \tau_z] = 2P_Y[\tau < \tau_a]$$

And it can be observed that

$$P_x|_{G_n^W}[\tau_{z_n} < \tau_a] = 2P_Y[\tau < \tau_a]$$

According Theorem 5 we have

$$P_x|_{G_n^W}[\tau_{z_n} < \tau_a] \leq \frac{R_n^W(x \leftrightarrow a \cup z_n)}{R_n^W(x \leftrightarrow z_n)}$$

where R_n denotes resistance on finite network G_n^W. Now according to Theorem 3

$$R_n^W(x \leftrightarrow a \cup z_n) = \min\left\{||\theta_n^W||_r^2;\ \theta_n^W \ is\ a\ flow\ from\ x \leftrightarrow a \cup z_n\right\}$$
$$R_n^W(x \leftrightarrow a) = \min\left\{||\alpha_n^W||_r^2;\ \alpha_n^W \ is\ a\ flow\ from\ x \leftrightarrow a \cup z_n\right\}$$

θ_n^W is a flow on G_n^W from x to $a \cup z_n$, and α_n^W is a flow from x to a. It can be observed that α_n^W is also a flow from x to $a \cup z_n$, so we have

$$R_n^W(x \leftrightarrow a \cup z_n) \leq R_n^W(x \leftrightarrow a)$$

And also

$$R_n(x \leftrightarrow a) = \min\left\{\sum_{e \in G_{n\frac{1}{2}}} r(e)\theta_n(e)^2\right\}$$

Here θ_n is a flow on G_n and it is naturally a flow on G_n^W(just let the flow on the edges in $G_n^W \backslash G_n$ be 0), so

$$R_n^W(x \leftrightarrow a) \leq R_n(x \leftrightarrow a)$$

Also, θ_n is a flow on G_{n+1}(just let the flow on the edges in $G_{n+1} \backslash G_n$ be 0), so

$$R_{n+1}(x \leftrightarrow a) \leq R_n(x \leftrightarrow a)$$

Because G is recurrent,

$$\lim_{n \to \infty} R_n^W(x \leftrightarrow z_n) = \infty$$

So

$$\lim_{n \to \infty} \frac{R_n^W(x \leftrightarrow a \cup z_n)}{R_n^W(x \leftrightarrow z_n)} \leq \lim_{n \to \infty} \frac{R_n(x \leftrightarrow a)}{R_n^W(x \leftrightarrow z_n)} \leq \lim_{n \to \infty} \frac{R_1(x \leftrightarrow a)}{R_n^W(x \leftrightarrow z_n)} = 0$$

That is

$$v(x) = \lim_{n \to \infty} v_n(x) = \lim_{n \to \infty} v_n(x) = \lim_{n \to \infty} P_x^n[\tau_a < \tau_z] = P_x[\tau_a < \tau_z] \qquad (8)$$

(b)Because i_n is a flow on G_n, it is naturally a flow on G_{n+1}(just let the flow on the edges in

$G_{n+1} \backslash G_n$ be 0), and it is obviously in $l_-^2(G)$(just let the flow on the edges in $G \backslash G_n$ be 0), so by Theorem 3, $||i_n||_r \geq ||i_{n+1}||_r$ and $||i_n||_r^2 = ||i_n - i_{n+1}||_r^2 + ||i_{n+1}||_r^2$.

So $||i_n||_r$ is a positive decreasing Cauchy sequence in $\mathbb{R}$, and thus $\{i_n\}$ is a Cauchy sequence in the Banach space $l_-^2(G)$, so it has a limit $i \in l_-^2(G)$.

(c)It is obvious that $\varepsilon(i_n) dv_n(e) = i_n r(e)$ is the Ohm's law. Because i_n and v_n have pointwise limit and $\varepsilon(i_n) = ||i_n||_r^2$ if i_n is regarded as a flow on G, so the limit situation is:

$$\varepsilon(i) dv = ir \tag{9}$$

Theorem 2.7[3] Let G_n^W be the graph obtained from G by identifying the vertices G_n into one vertex z_n, Let i_n^W be the unit current flow on G_n from a to z, then $i_n^W \to i$ in norm.

Theorem 2.6 and 2.7 shows that the resistance of two points on a sequence of exhaustion graphs converges to a limit, and it can be defined as the resistance between these two points on the infinite graph. When the infinite network is defined to be Z^2, many papers show the similar result[4] that the resistance between two nodes on Δ_N converges to a limit.

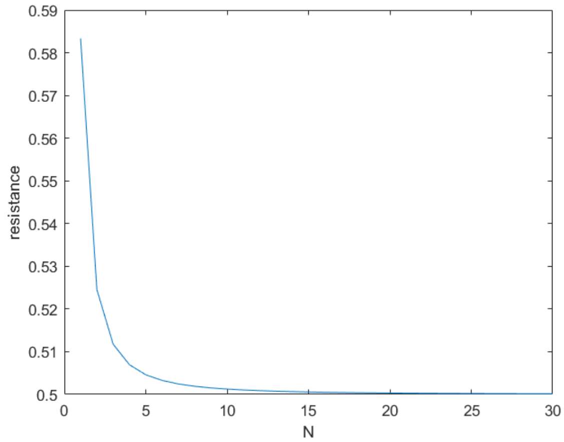

Fig.2 The Resistance between Point (0,0) and (0,1) on Δ_n Converges to 1/2

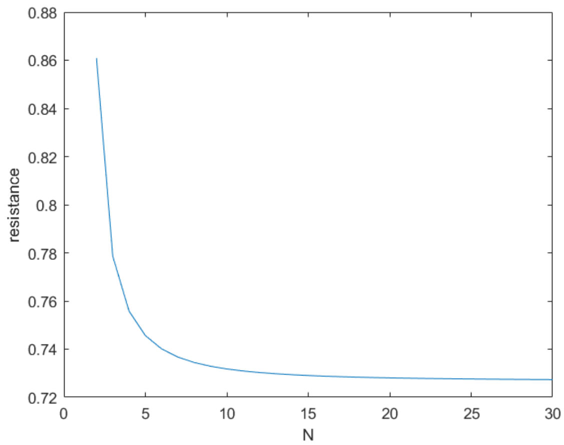

Fig.3 The Resistance between Point (0,0) and (0,2) on Δ_n Converges to 2-4/Π

3. The Voltage Function in $l^2(Z^2)$

Consider the Fourier transform:$l^2(Z^2) \to L^2([0,2\pi]^2)$

$$a \to \sum_{(m,n) \in Z^2} a(m,n) e^{-i(mx+ny)} \tag{10}$$

It is an isometry and many articles[5] use this fact to solve for solutions in function spaces $l^2(Z^2)$

H_{Z^2} is an operator on $l^2(Z^2)$ and thus can induce an operator on $L^2([0,2\pi]^2)$. The corresponding operator of H_{Z^2} on the space $L^2([0,2\pi]^2)$ is:

$$h: L^2([0,2\pi]^2) \to L^2([0,2\pi]^2)$$
$$f \longmapsto (-2cosx - 2cosy + 4)f$$

That is

$$\Psi\left(H_{Z^2}(v)\right) = (-2cosx - 2cosy + 4) * \Psi(\text{v}) \quad (11)$$

For a current flowing into point a=(m_1,n_1) and out of point z=(m_2,n_2) then the corresponding voltage function in $l^2(Z^2)$ is

$$v = \Psi^{-1}\left(\frac{e^{-i(m_1x+n_1y)} - e^{-i(m_2x+n_2y)}}{(-2cosx - 2cosy + 4)}\right) \quad (12)$$

Obviously,

$$\frac{e^{-i(m_1x+n_1y)} - e^{-i(m_2x+n_2y)}}{(-2cosx - 2cosy + 4)}$$
$$= \frac{(\cos(m_1x + n_1y) - \cos(m_2x + n_2y)) - i(\sin(m_1x + n_1y) - \sin(m_2x + n_2y))}{(-2cosx - 2cosy + 4)}$$
$$= \frac{O(|m_1 - m_2|x + |n_1 - n_2|y)}{O(x^2 + y^2)}$$

This function is not always locally L^2 integrable near the origin point (0,0).

This means H_{Z^2} is not invertible on $l^2(Z^2)$, which matches the result that 0 lies in the spectral of H_{Z^2}⑥. So, this method is not rigorous, though it has been widely used to calculate the voltage function on $Z^2$⑦. The following section shows that if the Hilbert space $l^2(Z^2)$ is appropriately changed into other spaces, the inverse of operator H_{Z^2} can be well-defined and be the limit of the pseudoinverse of the Laplacian matrices of a sequence of finite graphs.

4. The Pseudoinverse Operator in $D/\mathbb{R}$

Define the Dirichlet functions space on vertices of the graph:

$$D = \{f; cdf \in l^2_-(E,r)\} \quad (13)$$

Dirichlet functions space is a Hilbert space with the inner product:

$$< f, g \geq f(o)g(o) + (cdf, cdg)_r \quad (14)$$

Where o is an arbitrary point in the vertices set of the graph.

Obviously, the space $l^2(V)$ is a subspace of D, also the constant functions space $\mathbb{R}$ is a closed subspace in Dirichlet functions space, and there is a natural inner product in Hilbert space $D/\mathbb{R}$ inherited from D:

$$< f + \mathbb{R}, g + \mathbb{R} >= (cdf, cdg)_r$$

Theorem 4.1 If the Dirichlet function space is defined on network Z^2, the only harmonic function in D is constant[3].

Theorem 4.2 $S_1 = \{f+\mathbb{R} \in D/\mathbb{R}; f(a)=1, f(z)=-1, f=0 \text{ elsewhere}\}$ is dense in $D/\mathbb{R}$ and $l^2(Z^2)$.

Proof:

Consider $D^\perp$ in the Hilbert space D. Define the set of functions:

$$S_2=\{e_x \in D;\ e_x(x)=1, e_x=0 \text{ elsewhere}\}$$

Obviously, if the function $g\in S_1^{\perp}$ in $D/\mathbb{R}$, then:

$$< g+\mathbb{R}, e_a - e_z + \mathbb{R} >= 0$$

$$< g+\mathbb{R}, e_a + \mathbb{R} >=< g+\mathbb{R}, e_z + \mathbb{R} >$$

For any *a* and *z,* and because

$$< g+\mathbb{R}, g+\mathbb{R} >< \infty$$

So,

$$\lim_{|a|\to\infty} < g+\mathbb{R}, e_a + \mathbb{R} >= 0$$

Then

$$< g+\mathbb{R}, e_z + \mathbb{R} >= 0$$

For any *z* in the graph. So

$$g(z) = \sum_{y\ adjiacent\ to\ z} c(z,y)\, g(y)$$

g is a harmonic function in *D*, so

$$g \in \mathbb{R}$$

This means S_1 is dense in $D/\mathbb{R}$.
Consider the map:

$$H_{Z^2}: l^2(Z^2) \to l^2(Z^2)$$

Obviously, the domain of the operator can be extended to D, because:

$$||H_{Z^2}(f)||^2_{l^2(Z^2)} = \sum_{x\in Z^2}\left(\sum_{x,y\ adjacent}\big(f(x)-f(y)\big)\right)^2 \le ||f||_D^2$$

So H_{Z^2} is the continuous map from *D* to $l^2(Z^2)$. Because

$$H_{Z^2}(\mathbb{R}) = 0$$

We have a naturally defined bounded operator:

$$\widehat{H_{Z^2}}: D/\mathbb{R} \to l^2(Z^2)$$

$$\widehat{H_{Z^2}}(f+\mathbb{R}) = H_{Z^2}(f)$$

Also, we can extend the definition of the pseudoinverse $H^{\dagger}_{\Delta_N^w}$ so that it will be a bounded operator from S_1 to *D*. Assume that $f(a)=1$, $f(z)=-1$, $f=0$ elsewhere and the vertices *a* and *z* lies in Δ_N (Because Δ_N is an exhaustion of Z^2, so for any *f* there is a Δ_N large enough to contain *a* and *z*).

$$\widetilde{H^{\dagger}_{\Delta_N^w}}(f)(o) = \begin{cases} H^{\dagger}_{\Delta_N^w}(f)(o)\,;\, o \in \Delta_N \\ H^{\dagger}_{\Delta_N^w}(f)(z_n)\,;\, o \notin \Delta_N \end{cases}$$

Define

$$\widehat{H^{\dagger}_{\Delta_N^w}}: S_1 \to D/\mathbb{R}$$

$$\widehat{H^{\dagger}_{\Delta_N^w}}(f) = \widetilde{H^{\dagger}_{\Delta_N^w}}(f) + \mathbb{R}$$

So that

$$||\widehat{H^{\dagger}_{\Delta_N^w}}(f)||^2_{D/\mathbb{R}} = ||\widetilde{H^{\dagger}_{\Delta_N^w}}(f) + \mathbb{R}||^2_{D/\mathbb{R}}$$

$$= \sum_{x\in Z^2} \Big(\sum_{\substack{x,y\ adjacent \\ x,y\in\Delta_N}} \left(\widetilde{H^{\dagger}_{\Delta_N^w}}(f)(x) - \widetilde{H^{\dagger}_{\Delta_N^w}}(f)(y)\right)^2 + \sum_{\substack{x,y\ adjacent \\ x\in\Delta_N, y\notin\Delta_N}} \left(\widetilde{H^{\dagger}_{\Delta_N^w}}(f)(x) - \widetilde{H^{\dagger}_{\Delta_N^w}}(f)(z_n)\right)^2 \Big)$$

$$= \sum_{e\in\Delta_N^w} i_N^w(e)^2 = ||i_N^w||_r^2$$

Where i_N^w is a unit flow from a to z.

Theorem 4.3 the operator $\widehat{H^{\dagger}_{\Delta_N^w}}$ converges to $\widehat{H_{Z^2}}^{-1}$ in the strong operator topology from normed linear space spanned by S_1, which is dense in $l^2(Z^2)$, to the space $D/\mathbb{R}$.

Proof:

From theorem 2.6 and theorem 2.7, if $I\in S_1$, there is a limit voltage in D function on Z^2 that:

$$H_{Z^2}(v) = I$$

Because

$$||v||_D^2 = v(o)^2 + \sum_{x\in Z^2}\left(\sum_{x,y\ adjacent}\big(v(x) - v(y)\big)\right)^2 = v(o)^2 + ||cdv||_r^2 = v(o)^2 + ||i||_r^2 < \infty$$

So, $v \in D$, and for any other function in D that:

$$H_{Z^2}(v) = H_{Z^2}(u) = I$$

Then $u\text{-}v \in D$ and $u\text{-}v$ is harmonic. From theorem 4.1, we have:

$$u - v \in \mathbb{R}$$

Then $v+\mathbb{R}$ is the only function in $D/\mathbb{R}$ that satisfies:

$$\widehat{H_{Z^2}}(v + \mathbb{R}) = I$$

So, $\widehat{H_{Z^2}}^{-1}$ is well defined from S_1 to $D/\mathbb{R}$.

Then, from theorem 2.7 we get that for any function f in S_1

$$\lim_{n\to\infty} ||\widehat{H^{\dagger}_{\Delta_N^w}}(f) - \widehat{H_{Z^2}}^{-1}(f)||^2_{D/\mathbb{R}} = \lim_{n\to\infty} ||i_N^w - i||_r^2 = 0$$

So, the operator $\widehat{H^{\dagger}_{\Delta_N^w}}$ converges to $\widehat{H_{Z^2}}^{-1}$ in the strong operator topology from normed linear space spanned by S_1 to the space $D/\mathbb{R}$.

5. Conclusion

This paper gives a proper definition of the voltage function of a corresponding current flow and proves that there is a series of finite grids as an exhaustion of Z^2 that on this series of finite grids, the resistance between two given points converges to the resistance on infinite grid Z^2 calculated using voltage function defined in space $l^2(Z^2)$. This makes the physical image of the defined voltage function on Z^2 clear because it is the limit situation of finite grids. This result also shows that in a specially defined operator topology, the pseudoinverse operators of the Laplacians of a sequence of wired graphs converges.

Notations

Notations	Meaning
$l^2(Z^2)$	the L_2 space on Z^2 with the counting measure.
$l^2(\Delta)$	functions in $l^2(Z^2)$ that are supported in Δ
Δ_N	square centered at $(0,0)$ with side length $2N+1$
$\hat{e}_N$	function in $l^2(\Delta_N)$ where $\hat{e}_N(m)=\begin{cases}1; & m\in\Delta_N\\0; & m\notin\Delta_N\end{cases}$ Δ is a subset of Z^2
τ_x	the first time that a random walk reaches x
τ_x^+	the first positive time that a random walk starting at x reaches x
H_{Δ_N}	Laplacian matrix of graph Δ_N
$H_{\Delta_N}^{\dagger}$	pseudoinverse of Laplacian matrix of graph Δ_N
H_{Z^2}	graph Laplacian of Z^2

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